Lecture 11 – Parabolic Fixed Points.

Assume $f(0)=0$, $f'(0)=\lambda$ so that $\lambda^q=1$ for some q. Up to taking q^{th} iterate, assume $\lambda=1$

$$f(z) = z + a z^{n+1} + \dots \qquad (a\neq 0)$$

Theorem (Flower thm, Leau–Fatou)

If f is as above, there exist n attracting petals $\mathcal{U}_i$ and n repelling petals $\mathcal{U}_i'$ so that

$$\mathcal{U}_i \cap \mathcal{U}_j = \emptyset \qquad \text{if } i\neq j$$

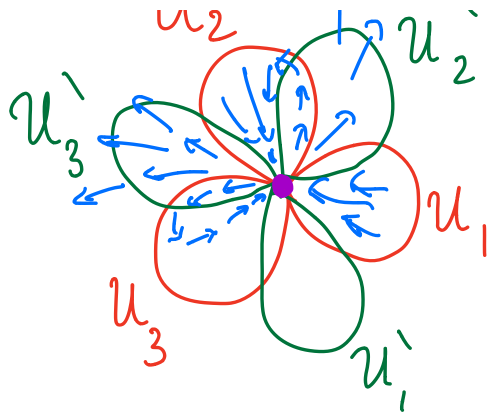
$$\mathcal{U}_i' \cap \mathcal{U}_j' = \emptyset \qquad \text{if } i\neq j$$

$$\mathcal{U}_i \cap \mathcal{U}_i' \neq \emptyset$$

$$\mathcal{U}_i \cap \mathcal{U}_{i+1}' \neq \emptyset$$

$$\mathcal{U}_i \cap \mathcal{U}_j' = \emptyset \qquad \text{if } j\neq i, i+1.$$

Moreover, $N_0 := \bigcup_{i=1}^{n} \mathcal{U}_i \cup \mathcal{U}_i' \cup \{0\}$ is a neighborhood of the origin.



Def.: A connected open set $\mathcal{U}$ is an attracting petal if

$$f(\overline{\mathcal{U}}) \subset \mathcal{U} \cup \{0\} \qquad \text{and}$$

$$\bigcap_{k \geq 0} f^k(\overline{\mathcal{U}}) = \{0\}.$$

A repelling petal is attracting petal for f^{-1}.

Rmk Attracting & Repelling petals are NOT canonical!

Cor.: In N_0 there is no periodic orbit other than zero.

Cor.: Any orbit which does not hit the fixed point converges to the fixed point iff it eventually lands in one of the attracting petals.

Cor.: Each basin of attraction of the fixed point is contained in the Fatou set, but the boundary of each basin is contained in the Julia set. In particular, the parabolic fixed point belongs to $J(f)$.

Pf.: $0 \in J(f)$: $\quad f(z) = z + a z^{n+1} + \dots$

$$f^k(z) = z + k a z^{n+1} + \dots$$

$\frac{d}{dz^{n+1}}\left(f^k(0)\right) \longrightarrow \infty \quad$ so $\left(f^k\big|_{\mathcal{U}}\right)$ is not normal for any $\mathcal{U} \ni 0$.

Let Ω_i be the basins of attraction of $z=0$. Then $f^k\big|_{\Omega_i} \to 0$, so $f^k\big|_{\Omega_i}$ is normal

If $z \in \partial\Omega_i$ then there exist $\varepsilon > 0$ and $(n_k) \to \infty$ s.t. $d(f^{n_k}(z), 0) > \varepsilon$

If $\mathcal{U} \ni z$ is open, then

$$f^{n_k}\big|_{\mathcal{U} \cap \Omega_i} \longrightarrow 0$$

So there pts close to z which converge to 0 but z stays away from it, hence $f^{n_k}\big|_{\mathcal{U}}$ is NOT normal

$\Rightarrow z \in J(f)$.

Proof of Flower Theorem

$$f(z) = z + a z^{n+1} + \dots$$

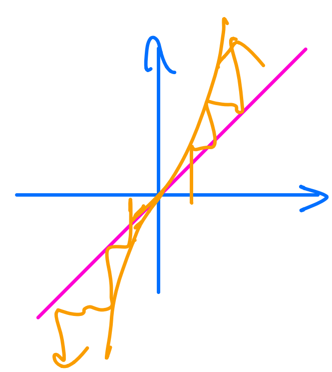
$\vec{v}$ is an attracting direction if

$$a v^n < 0$$

v is a repelling direction if

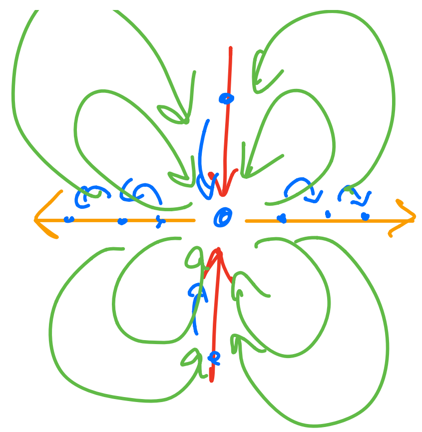
$$a v^n > 0$$

E.g.: $f(z) = z + z^3$



ATTR: $v^2 < 0$ $\quad v = \pm i$

REP: $v^2 > 0$ $\quad v = \pm 1$



$$f(i) = i - i = 0$$

$$f(-i) = -i + i = 0$$

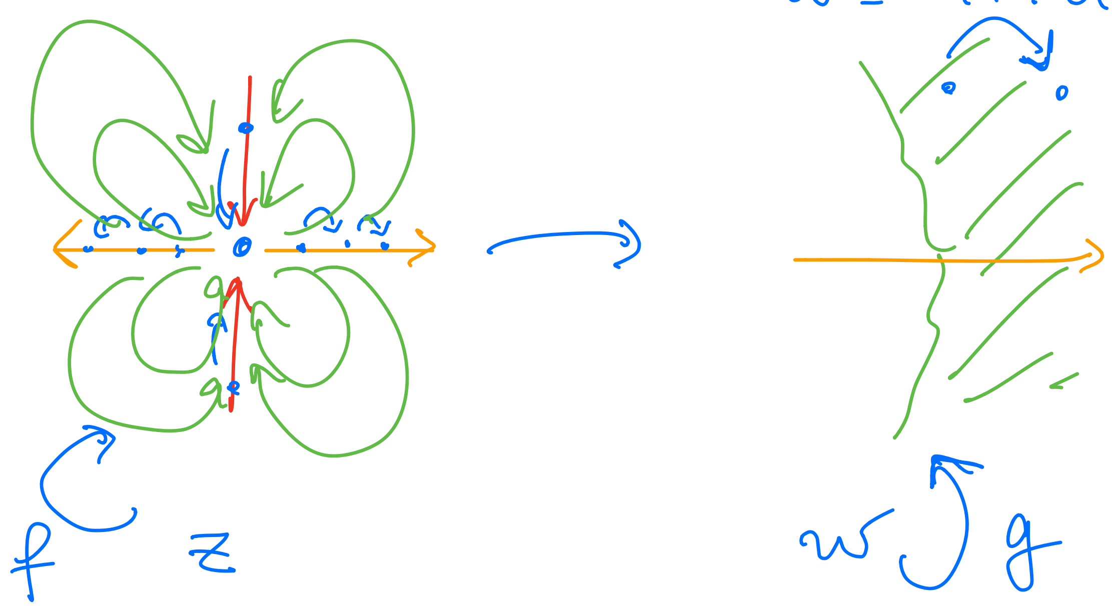
$$f(z) - z = z^3$$

$$z \mapsto z\left(1 + a z^n + o(z^n)\right)$$

$$z = \left(\frac{b}{w}\right)^{1/n} \qquad z^n = \frac{b}{w}$$

$$w = \frac{b}{z^n}$$

$$w' = w + 1 + o(1)$$



$$z \mapsto z\left(1 + a z^n + o(z^n)\right)$$

$$w' = g(w)$$

$$\| $$

$$\frac{b}{f(z)^n} = \frac{b}{z^n} \cdot \frac{1}{\left(1 + a z^n + o(z^n)\right)^n} =$$

$$w' = w \frac{1}{\left(1 + \frac{ab}{w} + o\left(\frac{b}{w}\right)\right)^n}$$

choose b s.t. $b = -\frac{1}{na}$

$$w' = w \frac{1}{\left(1 - \frac{1}{nw} + o\left(\frac{1}{w}\right)\right)^n}$$

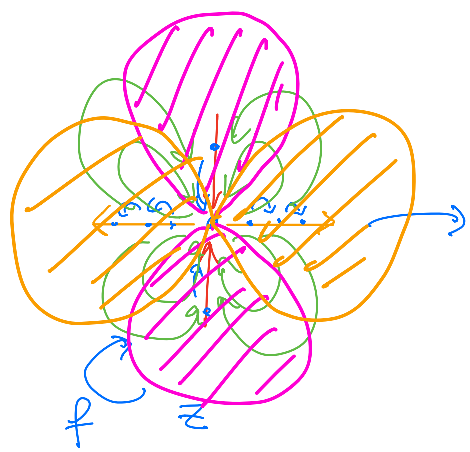
$$= w\left(1 - \frac{1}{nw} + o\left(\frac{1}{w}\right)\right)^{-n} =$$

$$= w\left(1 + \frac{1}{w} + o\left(\frac{1}{w}\right)\right)$$

$w' = w + 1 + o(1)$

Define petal (depends on r, ε, c)

$$P = \left\{ w = u + iv, \quad |w| > r, \quad u > c - \frac{|v|}{\tan(2\varepsilon)} \right\}$$



Since $g(w) = w + 1 + o(1)$, then choosing $c, r \gg 1$, $\varepsilon << 1$ we get

$$g(\overline{P}) \subset P$$

The coordinate w is called FATOU COORDINATE

Thm Let $\mathcal{U}$ be an attracting petal and define R. surface $\mathcal{U}/f$ by identifying $z \sim f(z)$. Then $\mathcal{U}/f$ is isomorphic to $\mathbb{C}/\mathbb{Z}$.

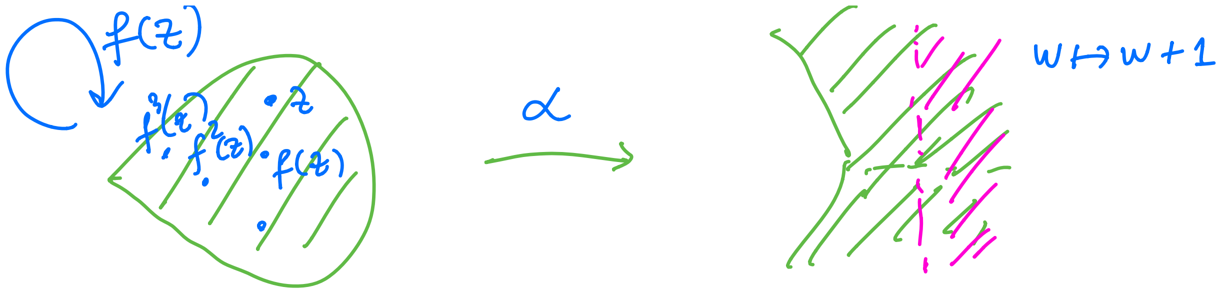
Moreover, there is a unique (up to translation) univalent embedding $\alpha : \mathcal{U} \longrightarrow \mathbb{C}$ which satisfies

$$\boxed{\alpha(f(z)) = 1 + \alpha(z)}$$

ABEL FUNCT. EQ.

for all $z \in \mathcal{U} \cap f^{-1}(\mathcal{U})$. With suitable choice of $\mathcal{U}$, $\alpha(\mathcal{U}) \subset \mathbb{C}$ contains a right half plane.

[Same holds for repelling petal by replacing "right" with "left"]



$$\begin{tikzcd} \mathcal{U} \arrow[r, "\alpha"', "(\text{injective})"] \arrow[d, "z \sim f(z)"'] & \mathbb{C} \arrow[d] \\ \mathcal{U}/f \arrow[r, "\sim"] & \mathbb{C}/\mathbb{Z} \end{tikzcd}$$

Écalle cylinder

Cor.: If $\mathcal{U}$ is attracting petal, then $\alpha : \mathcal{U} \to \mathbb{C}$ extends uniquely to a holo map defile on the attracting basin (not injective anymore) so that

$$\alpha(f(z)) = 1 + \alpha(z)$$

$$\begin{tikzcd} \text{basin } \Omega \arrow[r, "\alpha", "\text{holo}"'] & \mathbb{C} \\ \text{petal } \mathcal{U} \arrow[u, phantom, "\cup"] \arrow[r, "\alpha", "\text{injective}"'] \arrow[loop below, "f"] & \mathbb{C} \arrow[u, phantom, "\parallel"] \arrow[loop below, "z \mapsto z+1"] \end{tikzcd}$$

α is called FATOU COORDINATE

Cor.: Every parabolic basin contains a critical point

Pf.: if not, you can extend α to an isomorphism between Ω and $\mathbb{C}$, which is contradiction since Ω is hyperbolic.

Cor.: The number of attracting + parabolic cycles is at most $2d-2$ for a rat'l map of degree d.

INDIFFERENT FIXED POINTS WITH IRRATIONAL ANGLE

$$f(0)=0 \qquad f'(0) = e^{2\pi i\alpha}, \quad \alpha \in \mathbb{R}\setminus\mathbb{Q}$$

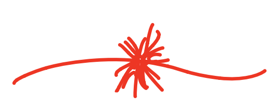
Def.: 0 is the center of a SIEGEL DISK if there exists a holo $U \xrightarrow{\varphi} \mathbb{D}$ and U a nbd of 0 s.t.

$$\varphi \circ f \circ \varphi^{-1}(z) = e^{2\pi i\alpha} z$$

$f(z) = e^{2\pi i\alpha} z$

Otherwise, 0 is a CREMER point.

"HEDGEHOGS"



Thm (Yoccoz)

Every fixed point of multiplier $e^{2\pi i\alpha}$ is center of Siegel disk if and only if α is a Brjuno number, i.e. if

$$\alpha = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}}$$ is the continued fraction expansion

and $\frac{p_n}{q_n} \to \alpha$ are the truncations,

$$\sum_{n=1}^{\infty} \frac{\log q_{n+1}}{q_n} < +\infty$$

Theorem

Let $f: \mathbb{C} \to \mathbb{C}$ be a rational map of degree $d \geq 2$. Then f has at most a finite number of cycles which are attracting or neutral. Indeed, the number of such cycles is $\leq 2d-2$.

Theorem

The Julia set of a rational map of degree $d \geq 2$ equals the closure of the set of repelling periodic pts.

Cor.: There are infinitely many repelling periodic points.